Riccati equations satisfied by
modular forms of level 5, 6 and 8

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Abstract In this paper, we derive differential equations of the Riccati type satisfied by modular forms of level 5, 6 and 8. For this purpose, we use the high-level versions of Jacobi's derivative formula.
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1 Introduction

There have been many researches in ordinary differential equations (ODEs) satisfied by modular forms. In particular, Hurwitz [1], Klein [4] and Van del Pol [9] considered the following ODE,

$$y'' + \frac{\pi^2}{36} E_4(\tau) y = 0, \tag{1.1}$$

where the Eisenstein series $E_2, E_4,$ and E_6 are defined by

$$E_2(q) = E_2(\tau) := 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}, \ E_4(q) = E_4(\tau) := 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n},$$

$$E_6(q) = E_4(\tau) := 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1-q^n}, \ q = \exp(2\pi i \tau), \tau \in \mathbb{H}^2,$$

and the upper half plane $\mathbb{H}^2$ is defined by

$$\mathbb{H}^2 = \{\tau \in \mathbb{C} \mid \Im \tau > 0\}.$$

Van del Pol [9] noticed that the ODE (1.1) corresponds to the Riccati equation,

$$\frac{6}{\pi i} u' + u^2 = E_4. \tag{1.2}$$

Thanks to Ramanujan's ODEs,

$$q\frac{E_2}{dq} = \frac{(E_2)^2 - E_4}{12}, \ q\frac{E_4}{dq} = \frac{E_2E_4 - E_6}{3}, \ q\frac{E_6}{dq} = \frac{E_2E_6 - (E_4)^2}{2},$$

$u = -E_2$ is a solution of the ODE (1.2). For the proof, see Sebbar et al. [10].

Sebbar et al. [10] studied the following Riccati equations,

$$\frac{k}{\pi i}u' + u^2 = E_4, \ (k = 1, 2, 3, 4, 5, 6),$$

and the correspoding second order ODE

$$y'' + \frac{\pi^2}{k^2}E_4(\tau)y = 0.$$

ODEs satisfied by modular forms also apper in mathematical physics. For example, Milas [7] derived the ODE of the form

$$y'' - 4\pi i E_2(\tau)y' + \frac{44}{5}\pi^2 E_4(\tau)y = 0, \tag{1.3}$$

from the character formula for the Virasoro model and the theory of vertex operator algebras. In particular, the solutions of the ODE (1.3) are given by modular forms of level 5.

The aim of this paper is to derive Riccati equations satisfied by modular forms of level 5, 6 and 8. In particular, we propose a new approach to ODEs satisfied by modular forms through the high-level versions of Jacobi's derivative formula, which were studied in [5] and [6].

This paper is organized as follows. In Section 2, we review the properties of the theta functions. In Section 3, we derive some theta-functional formulas. In Section 4, we recall the high-level versions of Jacobi's derivative formula.

In Section 5, we derive ODE satisfied by a modular form of level 4, which turns out to be equivalent to the formula of the number of representations of a natural number $n, (n = 1, 2, \ldots)$ as the sum of four triangular numbers. In Sections 6, 7, and 8, we deal with Riccati equations satisfied by modular forms of level 5, 6 and 8.

2 The properties of the theta functions

2.1 Notation

Throughout this paper, let $\mathbb{N}_0$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{C}^*$ denote the set of nonnegative integers, positive integers, integers, rational numbers, real numbers, complex numbers and nonzero complex numbers, respectively.

and the Dedekind eta function is defined by

$$\eta(\tau) = q^{\frac{1}{24}}(q;q)_\infty, \ (q;q)_\infty = \prod_{n=1}^{\infty}(1 - q^n), \ q = \exp(2\pi i\tau), \ \tau \in \mathbb{H}^2.$$

Following Farkas and Kra [2], we first introduce the theta function with characteristics, which is defined by

$$\theta\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}(\zeta,\tau) = \theta\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}(\zeta) := \sum_{n\in\mathbb{Z}}\exp\left(2\pi i\left[\frac{1}{2}\left(n+\frac{\epsilon}{2}\right)^2\tau + \left(n+\frac{\epsilon}{2}\right)\left(\zeta+\frac{\epsilon'}{2}\right)\right]\right)$$

$$= \sum_{n\in\mathbb{Z}}q^{\frac{1}{2}\left(n+\frac{\epsilon}{2}\right)^2}e^{2\pi i\left(n+\frac{\epsilon}{2}\right)\left(\zeta+\frac{\epsilon'}{2}\right)},$$

where $\epsilon, \epsilon' \in \mathbb{R}$, $\zeta \in \mathbb{C}$, $\tau \in \mathbb{H}^2$ and $q = \exp(2\pi i\tau)$. The theta constants are given by

$$\theta\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} := \theta\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}(0,\tau).$$

Let us denote the derivative coefficients of the theta functions by

$$\theta'\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} := \frac{\partial}{\partial\zeta}\theta\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}(\zeta,\tau)\bigg|_{\zeta=0}, \ \theta''\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} := \frac{\partial^2}{\partial\zeta^2}\theta\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}(\zeta,\tau)\bigg|_{\zeta=0}.$$

Jacobi's derivative formula is given by

$$\theta'\begin{bmatrix} 1 \\ 1 \end{bmatrix} = -\pi\theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}\theta\begin{bmatrix} 1 \\ 0 \end{bmatrix}\theta\begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{2.1}$$

2.2 Basic properties

We first note that for $m, n \in \mathbb{Z}$,

$$\theta\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}(\zeta + n + m\tau, \tau) = \exp(2\pi i)\left[\frac{n\epsilon - m\epsilon'}{2} - m\zeta - \frac{m^2\tau}{2}\right]\theta\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}(\zeta,\tau), \tag{2.2}$$

and

$$\theta\begin{bmatrix} \epsilon + 2m \\ \epsilon' + 2n \end{bmatrix}(\zeta,\tau) = \exp(\pi i\epsilon n)\theta\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}(\zeta,\tau). \tag{2.3}$$

Furthermore, it is easy to see that

$$\theta\begin{bmatrix} -\epsilon \\ -\epsilon' \end{bmatrix}(\zeta,\tau) = \theta\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}(-\zeta,\tau) \text{ and } \theta'\begin{bmatrix} -\epsilon \\ -\epsilon' \end{bmatrix}(\zeta,\tau) = -\theta'\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}(-\zeta,\tau).$$

3

For $m, n \in \mathbb{R}$, we see that

$$\theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} \left(\zeta + \frac{n + m\tau}{2}, \tau \right)$$

$$= \exp(2\pi i) \left[-\frac{m\zeta}{2} - \frac{m^2\tau}{8} - \frac{m(\epsilon' + n)}{4} \right] \theta \begin{bmatrix} \epsilon + m \\ \epsilon' + n \end{bmatrix} (\zeta, \tau). \qquad (2.4)$$

We note that $\theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\zeta, \tau)$ has only one zero in the fundamental parallelogram, which is given by

$$\zeta = \frac{1 - \epsilon}{2} \tau + \frac{1 - \epsilon'}{2}.$$

2.3 Jacobi's triple product identity

All the theta functions have infinite product expansions, which are given by

$$\theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\zeta, \tau) = \exp \left(\frac{\pi i \epsilon \epsilon'}{2} \right) x^{\frac{\epsilon^2}{4}} z^{\frac{\epsilon}{2}}$$

$$\times \prod_{n=1}^{\infty} (1 - x^{2n})(1 + e^{\pi i \epsilon'} x^{2n-1+\epsilon} z)(1 + e^{-\pi i \epsilon'} x^{2n-1-\epsilon}/z), \qquad (2.5)$$

where $x = \exp(\pi i \tau)$ and $z = \exp(2\pi i \zeta)$. Therefore, it follows from Jacobi's derivative formula (2.1) that

$$\theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix} (0, \tau) = -2\pi q^{\frac{1}{8}} \prod_{n=1}^{\infty} (1 - q^n)^3, \quad q = \exp(2\pi i \tau).$$

2.4 Spaces of Nth order θ-functions

Following Farkas and Kra [2], we define $\mathcal{F}_N \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}$ to be the set of entire functions f that satisfy the two functional equations

$$f(\zeta + 1) = \exp(\pi i \epsilon) \, f(\zeta)$$

and

$$f(\zeta + \tau) = \exp(-\pi i)[\epsilon' + 2N\zeta + N\tau] \, f(\zeta), \quad \zeta \in \mathbb{C}, \ \tau \in \mathbb{H}^2,$$

4

where N is a positive integer and $\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} \in \mathbb{R}^2$. This set of functions is called the space of Nth order θ-functions with characteristic $\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}$. Note that

$$\dim \mathcal{F}_N \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} = N.$$

For its proof, see Farkas and Kra [2, p.133].

2.5   The heat equation

The theta function satisfies the following heat equation:

$$\frac{\partial^2}{\partial \zeta^2} \theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\zeta, \tau) = 4\pi i \frac{\partial}{\partial \tau} \theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\zeta, \tau). \tag{2.6}$$

2.6   Lemma of Farkas and Kra

We recall the lemma of Farkas and Kra [2, p. 78].

Lemma 2.1. *For all characteristics* $\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}, \begin{bmatrix} \delta \\ \delta' \end{bmatrix}$ *and all* $\tau \in \mathbb{H}^2$, *we have*

$$\theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (0, \tau) \theta \begin{bmatrix} \delta \\ \delta' \end{bmatrix} (0, \tau)$$

$$= \theta \begin{bmatrix} \frac{\epsilon+\delta}{2} \\ \epsilon' + \delta' \end{bmatrix} (0, 2\tau) \theta \begin{bmatrix} \frac{\epsilon-\delta}{2} \\ \epsilon' - \delta' \end{bmatrix} (0, 2\tau) + \theta \begin{bmatrix} \frac{\epsilon+\delta}{2} + 1 \\ \epsilon' + \delta' \end{bmatrix} (0, 2\tau) \theta \begin{bmatrix} \frac{\epsilon-\delta}{2} + 1 \\ \epsilon' - \delta' \end{bmatrix} (0, 2\tau).$$

3   Theta-functional formula

Proposition 3.1. *For every* $(z, \tau) \in \mathbb{C} \times \mathbb{H}^2$, *we have*

$$\theta^2 \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} \theta^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} (z, \tau) + \theta^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \theta \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} (z, \tau) \theta \begin{bmatrix} 1 \\ \frac{3}{2} \end{bmatrix} (z, \tau) - \theta^2 \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} \theta^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} (z, \tau) = 0, \tag{3.1}$$

$$\theta^2 \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix} \theta \begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix} (z, \tau) \theta \begin{bmatrix} 1 \\ \frac{5}{3} \end{bmatrix} (z, \tau) - \theta^2 \begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix} \theta \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix} (z, \tau) \theta \begin{bmatrix} 1 \\ \frac{4}{3} \end{bmatrix} (z, \tau)$$

$$+ \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} \theta \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix} \theta^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} (z, \tau) = 0, \tag{3.2}$$

5

$$\theta^2 \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} \theta \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} (z,\tau) \theta \begin{bmatrix} 1 \\ \frac{7}{4} \end{bmatrix} (z,\tau) - \theta^2 \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} \theta \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} (z,\tau) \theta \begin{bmatrix} 1 \\ \frac{5}{4} \end{bmatrix} (z,\tau)$$
$$+ \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} \theta \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} \theta^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} (z,\tau) = 0, \tag{3.3}$$

$$\theta^2 \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix} \theta \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix} (z,\tau) \theta \begin{bmatrix} 1 \\ \frac{9}{5} \end{bmatrix} (z,\tau) - \theta^2 \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix} \theta \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix} (z,\tau) \theta \begin{bmatrix} 1 \\ \frac{7}{5} \end{bmatrix} (z,\tau)$$
$$+ \theta \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix} \theta \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix} \theta^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} (z,\tau) = 0. \tag{3.4}$$

Proof. We prove equation (3.1). The others can be proved in the same way. We first note that $\dim \mathcal{F}_2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 2$ and

$$\theta^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} (z,\tau), \ \theta \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} (z,\tau) \theta \begin{bmatrix} 1 \\ \frac{3}{2} \end{bmatrix} (z,\tau), \ \theta^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} (z,\tau) \in \mathcal{F}_2 \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Therefore, there exist some complex numbers x_1, x_2, and x_3, not all zero, such that

$$x_1 \theta^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} (z,\tau) + x_2 \theta \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} (z,\tau) \theta \begin{bmatrix} 1 \\ \frac{3}{2} \end{bmatrix} (z,\tau) + x_3 \theta^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} (z,\tau) = 0.$$

Note that in the fundamental parallelogram, the zero of $\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (z)$, $\theta \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} (z)$, or $\theta \begin{bmatrix} 1 \\ 1 \end{bmatrix} (z)$ is $z = 1/2, 1/4$, or 0. Substituting $z = 1/2, 1/4$, and 0, we have

$$x_2 \theta^2 \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} + x_3 \theta^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0,$$
$$x_1 \qquad\qquad + x_3 \qquad\qquad = 0,$$
$$x_1 \theta^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} - x_2 \theta^2 \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} \qquad\qquad = 0.$$

Solving this system of equations, we have

$$(x_1, x_2, x_3) = \alpha \left(\theta^2 \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}, \theta^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}, -\theta^2 \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} \right) \quad \text{for some } \alpha \in \mathbb{C} \setminus \{0\},$$

which proves the proposition. $\qquad\square$

4 High-level versions of Jacobi's derivative formula

From Matsuda [5, 6], recall the following derivative formulas.

Theorem 4.1. *For every $\tau \in \mathbb{H}^2$, we have*

$$\theta' \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} (0, \tau) = -\pi \theta^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 2\tau) \theta \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} (0, \tau), \tag{4.1}$$

$$\frac{\theta' \begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix}}{\theta \begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix}} = \frac{1}{6} \theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{\left(\theta^4 \begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix} - 3\theta^4 \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix} \right)}{\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} \theta \begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix} \theta^3 \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix}}, \quad \frac{\theta' \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix}}{\theta \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix}} = \frac{1}{3} \theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{\theta^4 \begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix}}{\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} \theta \begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix} \theta^3 \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix}}, \tag{4.2}$$

$$\frac{\theta' \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} (0, \tau)}{\theta \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} (0, \tau)} = -\frac{\pi}{2} \cdot \left\{ \theta^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 2\tau) - \theta^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} (0, 2\tau) \frac{\theta^2 \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} (0, \tau)}{\theta^2 \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} (0, \tau)} \right\}, \tag{4.3}$$

$$\frac{\theta' \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} (0, \tau)}{\theta \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} (0, \tau)} = -\frac{\pi}{2} \cdot \left\{ -\theta^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 2\tau) + \theta^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} (0, 2\tau) \frac{\theta^2 \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} (0, \tau)}{\theta^2 \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} (0, \tau)} \right\}, \tag{4.4}$$

$$\frac{\theta' \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix}}{\theta \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix}} = \theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{\theta^5 \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix} - 3\theta^5 \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix}}{10\theta^3 \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix} \theta^3 \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix}}, \quad \frac{\theta' \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix}}{\theta \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix}} = \theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{3\theta^5 \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix} + \theta^5 \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix}}{10\theta^3 \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix} \theta^3 \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix}}. \tag{4.5}$$

5 ODE satisfied by a modular form of level 4

5.1 Preliminary result

Proposition 5.1. *For every $\tau \in \mathbb{H}^2$, we have*

$$\frac{\theta'' \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, \tau)}{\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, \tau)} - \frac{\theta'' \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} (0, \tau)}{\theta \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} (0, \tau)} = -\pi^2 \theta^4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, 2\tau).$$

Proof. Comparing the coefficients of the term z^2 in equation (3.1), we have

$$\frac{\theta''\begin{bmatrix}1\\0\end{bmatrix}}{\theta\begin{bmatrix}1\\0\end{bmatrix}} - \frac{\theta''\begin{bmatrix}1\\\frac{1}{2}\end{bmatrix}}{\theta\begin{bmatrix}1\\\frac{1}{2}\end{bmatrix}} = \frac{\left\{\theta'\begin{bmatrix}1\\1\end{bmatrix}\right\}^2}{\theta^2\begin{bmatrix}1\\0\end{bmatrix}} - \left\{\frac{\theta'\begin{bmatrix}1\\\frac{1}{2}\end{bmatrix}}{\theta\begin{bmatrix}1\\\frac{1}{2}\end{bmatrix}}\right\}^2.$$

From Jacobi's derivative formula (2.1) and equation (4.1), we obtain

$$\frac{\theta''\begin{bmatrix}1\\0\end{bmatrix}}{\theta\begin{bmatrix}1\\0\end{bmatrix}} - \frac{\theta''\begin{bmatrix}1\\\frac{1}{2}\end{bmatrix}}{\theta\begin{bmatrix}1\\\frac{1}{2}\end{bmatrix}} = -\pi^2\left(\theta^4\begin{bmatrix}0\\0\end{bmatrix}(0,2\tau) - \theta^2\begin{bmatrix}0\\0\end{bmatrix}(0,\tau)\theta^2\begin{bmatrix}0\\1\end{bmatrix}(0,\tau)\right)$$

$$= -\pi^2\left(\theta^4\begin{bmatrix}0\\0\end{bmatrix}(0,2\tau) - \theta^4\begin{bmatrix}0\\1\end{bmatrix}(0,2\tau)\right) \text{ by Lemma 2.1.}$$

Recall Jacobi's quartic identity:

$$\theta^4\begin{bmatrix}0\\0\end{bmatrix}(0,\tau) = \theta^4\begin{bmatrix}0\\1\end{bmatrix}(0,\tau) + \theta^4\begin{bmatrix}1\\0\end{bmatrix}(0,\tau).$$

For the proof, see Farkas and Kra [2, p. 120].

Therefore, it follows that

$$\frac{\theta''\begin{bmatrix}1\\0\end{bmatrix}(0,\tau)}{\theta\begin{bmatrix}1\\0\end{bmatrix}(0,\tau)} - \frac{\theta''\begin{bmatrix}1\\\frac{1}{2}\end{bmatrix}(0,\tau)}{\theta\begin{bmatrix}1\\\frac{1}{2}\end{bmatrix}(0,\tau)} = -\pi^2\theta^4\begin{bmatrix}1\\0\end{bmatrix}(0,2\tau).$$

$\square$

Theorem 5.2. *For each $n \in \mathbb{N}_0$,*

$$t_4(n) := \sharp\left\{(x,y,z,w) \in \mathbb{N}_0^4 \,\middle|\, n = \frac{x(x+1)}{2} + \frac{y(y+1)}{2} + \frac{z(z+1)}{2} + \frac{w(w+1)}{2}\right\}$$

$$= \sigma(2n+1),$$

where $\sigma(m)$ is the sum of positive divisors of $m \in \mathbb{N}$.

Proof. Set $q = \exp(2\pi i\tau)$. Jacobi's triple product identity (2.5) implies

$$\frac{\theta''\begin{bmatrix} 1 \\ 0 \end{bmatrix}(0,\tau)}{\theta\begin{bmatrix} 1 \\ 0 \end{bmatrix}(0,\tau)} - \frac{\theta''\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}(0,\tau)}{\theta\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}(0,\tau)} = -8\pi^2 \sum_{N=1}^{\infty} \left\{ 2\sigma(N) - 6\sigma\left(\frac{N}{2}\right) + 4\sigma\left(\frac{N}{4}\right) \right\} q^N$$

$$= -16\pi^2 \sum_{n=0}^{\infty} \sigma(2n+1)q^{2n+1}.$$

From the definition, it follows that

$$-\pi^2\theta^4 \begin{bmatrix} 1 \\ 0 \end{bmatrix}(0,2\tau) = -\pi^2 \left\{ \sum_{n\in\mathbb{Z}} q^{\left(n+\frac{1}{2}\right)^2} \right\}^4 = -16\pi^2 q \left\{ \sum_{n=0}^{\infty} q^{2\cdot\frac{n(n+1)}{2}} \right\}^4$$

$$= -16\pi^2 \sum_{n=0}^{\infty} t_4(n)q^{2n+1}.$$

The theorem follows from Proposition 5.1. $\qquad\square$

5.2 Derivation of a differential equation

Theorem 5.3. *For every $\tau \in \mathbb{H}^2$, we have*

$$\frac{d}{d\tau}\left\{ \frac{\eta^3(2\tau)}{\eta^2(\tau)\eta(4\tau)} \right\} = 4\pi i \frac{\eta^7(4\tau)}{\eta^2(\tau)\eta(2\tau)}, \tag{5.1}$$

that is,

$$\frac{d}{dq}\left\{ \frac{(q^2;q^2)_\infty^3}{(q;q)_\infty^2(q^4;q^4)_\infty} \right\} = 2\frac{(q^4;q^4)_\infty^7}{(q;q)_\infty^2(q^2;q^2)_\infty}, \quad q = \exp(2\pi i\tau).$$

Proof. The heat equation (2.6) and Proposition 5.1 imply that

$$4\pi i \left(\frac{\frac{d}{d\tau}\theta\begin{bmatrix} 1 \\ 0 \end{bmatrix}(0,\tau)}{\theta\begin{bmatrix} 1 \\ 0 \end{bmatrix}(0,\tau)} - \frac{\frac{d}{d\tau}\theta\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}(0,\tau)}{\theta\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}(0,\tau)} \right) = -\pi^2\theta^4\begin{bmatrix} 1 \\ 0 \end{bmatrix}(0,2\tau).$$

Multiplying $\theta\begin{bmatrix} 1 \\ 0 \end{bmatrix}(0,\tau)/\theta\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}(0,\tau)$ by both sides, we have

$$4\pi i \frac{d}{d\tau}\left\{ \frac{\theta\begin{bmatrix} 1 \\ 0 \end{bmatrix}(0,\tau)}{\theta\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}(0,\tau)} \right\} = -\pi^2\theta^4\begin{bmatrix} 1 \\ 0 \end{bmatrix}(0,2\tau) \cdot \frac{\theta\begin{bmatrix} 1 \\ 0 \end{bmatrix}(0,\tau)}{\theta\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}(0,\tau)}.$$

9

Jacobi's triple product identity (2.5) shows

$$\frac{\theta\begin{bmatrix}1\\0\end{bmatrix}(0,\tau)}{\theta\begin{bmatrix}1\\\frac{1}{2}\end{bmatrix}(0,\tau)} = \sqrt{2}\prod_{n=1}^{\infty}\frac{(1-q^{2n})^3}{(1-q^n)^2(1-q^{4n})}$$

and

$$\theta^4\begin{bmatrix}1\\0\end{bmatrix}(0,2\tau)\cdot\frac{\theta\begin{bmatrix}1\\0\end{bmatrix}(0,\tau)}{\theta\begin{bmatrix}1\\\frac{1}{2}\end{bmatrix}(0,\tau)} = 16\sqrt{2}q\prod_{n=1}^{\infty}\frac{(1-q^{4n})^7}{(1-q^n)^2(1-q^{2n})},$$

which proves the theorem. $\square$

Remark

Theorems 5.2 and 5.3 follow from Proposition 5.1. Therefore, the differential equation (5.1) is equivalent to the formula of $t_4(n)$, the number of representations of n as the sum of four triangular numbers.

6 ODE satisfied by a modular form of level 5

6.1 Preliminary results

Proposition 6.1. *For every $\tau \in \mathbb{H}^2$, we have*

$$\frac{\theta''\begin{bmatrix}1\\\frac{1}{5}\end{bmatrix}}{\theta\begin{bmatrix}1\\\frac{1}{5}\end{bmatrix}} - \frac{\theta''\begin{bmatrix}1\\\frac{3}{5}\end{bmatrix}}{\theta\begin{bmatrix}1\\\frac{3}{5}\end{bmatrix}} = \frac{\left\{\theta'\begin{bmatrix}1\\1\end{bmatrix}\right\}^2}{100\theta^6\begin{bmatrix}1\\\frac{1}{5}\end{bmatrix}\theta^6\begin{bmatrix}1\\\frac{3}{5}\end{bmatrix}}\left\{-8\theta^{10}\begin{bmatrix}1\\\frac{1}{5}\end{bmatrix} + 88\theta^5\begin{bmatrix}1\\\frac{1}{5}\end{bmatrix}\theta^5\begin{bmatrix}1\\\frac{3}{5}\end{bmatrix} + 8\theta^{10}\begin{bmatrix}1\\\frac{3}{5}\end{bmatrix}\right\}.$$

Proof. The proposition can be obtained by considering equation (4.5) and comparing the coefficients of the term z^2 in equation (3.4). $\square$

6.2 Derivation of a Riccati equation

Theorem 6.2. *For every $\tau \in \mathbb{H}^2$, set*

$$W = \theta^5\begin{bmatrix}1\\\frac{1}{5}\end{bmatrix}(0,\tau)/\theta^5\begin{bmatrix}1\\\frac{3}{5}\end{bmatrix}(0,\tau), \quad and \ q = \exp(2\pi i\tau).$$

Then, W satisfies the following Riccati equation:

$$q\frac{d}{dq}W = \frac{1}{(\sqrt{5})^3}\frac{(q;q)_\infty^5}{(q^5;q^5)_\infty}\left(W^2 - 11W - 1\right).\tag{6.1}$$

Proof. The heat equation (2.6) and Proposition 6.1 imply that

$$4\pi i\frac{d}{d\tau}\log\theta\begin{bmatrix}1\\\frac{1}{5}\end{bmatrix} - 4\pi i\frac{d}{d\tau}\log\theta\begin{bmatrix}1\\\frac{3}{5}\end{bmatrix}$$

$$=\frac{\left\{\theta'\begin{bmatrix}1\\1\end{bmatrix}\right\}^2}{100\,\theta\begin{bmatrix}1\\\frac{1}{5}\end{bmatrix}\theta\begin{bmatrix}1\\\frac{3}{5}\end{bmatrix}}\frac{\left\{-8\theta^{10}\begin{bmatrix}1\\\frac{1}{5}\end{bmatrix} + 88\theta^5\begin{bmatrix}1\\\frac{1}{5}\end{bmatrix}\theta^5\begin{bmatrix}1\\\frac{3}{5}\end{bmatrix} + 8\theta^{10}\begin{bmatrix}1\\\frac{3}{5}\end{bmatrix}\right\}}{\theta^5\begin{bmatrix}1\\\frac{1}{5}\end{bmatrix}\theta^5\begin{bmatrix}1\\\frac{3}{5}\end{bmatrix}},$$

which shows that

$$4\pi i\frac{d}{d\tau}\log\theta^5\begin{bmatrix}1\\\frac{1}{5}\end{bmatrix} - 4\pi i\frac{d}{d\tau}\log\theta^5\begin{bmatrix}1\\\frac{3}{5}\end{bmatrix}$$

$$=\frac{\left\{\theta'\begin{bmatrix}1\\1\end{bmatrix}\right\}^2}{20\,\theta\begin{bmatrix}1\\\frac{1}{5}\end{bmatrix}\theta\begin{bmatrix}1\\\frac{3}{5}\end{bmatrix}}\frac{\left\{-8\theta^{10}\begin{bmatrix}1\\\frac{1}{5}\end{bmatrix} + 88\theta^5\begin{bmatrix}1\\\frac{1}{5}\end{bmatrix}\theta^5\begin{bmatrix}1\\\frac{3}{5}\end{bmatrix} + 8\theta^{10}\begin{bmatrix}1\\\frac{3}{5}\end{bmatrix}\right\}}{\theta^5\begin{bmatrix}1\\\frac{1}{5}\end{bmatrix}\theta^5\begin{bmatrix}1\\\frac{3}{5}\end{bmatrix}}.$$

Jacobi's triple product identity (2.5) yields

$$\frac{\left\{\theta'\begin{bmatrix}1\\1\end{bmatrix}\right\}^2}{\theta\begin{bmatrix}1\\\frac{1}{5}\end{bmatrix}\theta\begin{bmatrix}1\\\frac{3}{5}\end{bmatrix}} = \frac{4\pi^2}{\sqrt{5}}\prod_{n=1}^{\infty}\frac{(1-q^n)^5}{(1-q^{5n})} = \frac{4\pi^2}{\sqrt{5}}\frac{\eta(\tau)^5}{\eta(5\tau)}.$$

Setting

$$(X,Y) = \left(\theta^5\begin{bmatrix}1\\\frac{1}{5}\end{bmatrix},\ \theta^5\begin{bmatrix}1\\\frac{3}{5}\end{bmatrix}\right),$$

we obtain

$$\frac{dX}{d\tau}Y - X\frac{dY}{d\tau} = \frac{2\pi i}{(\sqrt{5})^3}\frac{\eta^5(\tau)}{\eta(5\tau)}\left(X^2 - 11XY - Y^2\right),$$

which implies that

$$\frac{d}{d\tau}W = \frac{d}{d\tau}\left(\frac{X}{Y}\right) = \frac{\frac{dX}{d\tau}Y - X\frac{dY}{d\tau}}{Y^2} = \frac{2\pi i}{(\sqrt{5})^3}\frac{\eta^5(\tau)}{\eta(5\tau)}\left(W^2 - 11W - 1\right).$$

The theorem can be obtained by considering that $\dfrac{d}{d\tau} = \dfrac{dq}{d\tau}\dfrac{d}{dq} = 2\pi i q\dfrac{d}{dq}$. $\qquad\square$

11

7 ODE satisfied by a modular form of level 6

7.1 Preliminary results

Proposition 7.1. *For every $\tau \in \mathbb{H}^2$, we have*

$$\theta^3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \theta \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix} - \theta^4 \begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix} + \theta^4 \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix} = 0.$$

Proof. We note that $\mathcal{F}_2 \begin{bmatrix} \frac{4}{3} \\ 0 \end{bmatrix} = 2$ and

$$\theta \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} (\zeta, \tau) \theta \begin{bmatrix} \frac{2}{3} \\ \frac{5}{3} \end{bmatrix} (\zeta, \tau), \quad \theta \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \end{bmatrix} (\zeta, \tau) \theta \begin{bmatrix} \frac{2}{3} \\ \frac{4}{3} \end{bmatrix} (\zeta, \tau),$$

$$\theta^2 \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} (\zeta, \tau), \qquad \qquad \theta^2 \begin{bmatrix} \frac{2}{3} \\ 0 \end{bmatrix} (\zeta, \tau) \in \mathcal{F}_2 \begin{bmatrix} \frac{4}{3} \\ 0 \end{bmatrix}.$$

Substituting

$$\zeta = \frac{\tau \pm 2}{6}, \ \frac{\tau \pm 1}{6}, \ \frac{\tau \pm 1}{6}, \ \frac{\tau + 3}{6},$$

we obtain

$$A\mathbf{x} = \mathbf{0}, \tag{7.1}$$

where

$$A = \begin{pmatrix} 0 & \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} \theta \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix} & \theta^2 \begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix} & \zeta_6 \theta^2 \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix} \\ -\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} \theta \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix} & 0 & \theta^2 \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix} & \zeta_6 \theta^2 \begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix} \\ -\theta^2 \begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix} & -\theta^2 \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix} & 0 & \zeta_6 \theta^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ -\theta^2 \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix} & -\theta^2 \begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix} & -\theta^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} & 0 \end{pmatrix},$$

$\zeta_6 = \exp(2\pi i/6)$, and $\mathbf{x} = {}^t(x_1, x_2, x_3, x_4)$.

Since the system of equations (7.1) has a nontrivial solution $\mathbf{x} \neq \mathbf{0}$, we have $\det A = 0$.

Therefore, it follows that

$$
\begin{vmatrix}
0 & \theta\begin{bmatrix}1\\0\end{bmatrix}\theta\begin{bmatrix}1\\\frac{2}{3}\end{bmatrix} & \theta^2\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix} & \theta^2\begin{bmatrix}1\\\frac{2}{3}\end{bmatrix} \\
-\theta\begin{bmatrix}1\\0\end{bmatrix}\theta\begin{bmatrix}1\\\frac{2}{3}\end{bmatrix} & 0 & \theta^2\begin{bmatrix}1\\\frac{2}{3}\end{bmatrix} & \theta^2\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix} \\
-\theta^2\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix} & -\theta^2\begin{bmatrix}1\\\frac{2}{3}\end{bmatrix} & 0 & \theta^2\begin{bmatrix}1\\0\end{bmatrix} \\
-\theta^2\begin{bmatrix}1\\\frac{2}{3}\end{bmatrix} & -\theta^2\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix} & -\theta^2\begin{bmatrix}1\\0\end{bmatrix} & 0
\end{vmatrix}
$$

$$
= \left(\theta^3\begin{bmatrix}1\\0\end{bmatrix}\theta\begin{bmatrix}1\\\frac{2}{3}\end{bmatrix} - \theta^4\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix} + \theta^4\begin{bmatrix}1\\\frac{2}{3}\end{bmatrix} \right)^2
$$

$$
= 0.
$$

$\square$

Proposition 7.2. *For every $\tau \in \mathbb{H}^2$, we have*

$$
\frac{\theta''\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}}{\theta\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}} - \frac{\theta''\begin{bmatrix}1\\\frac{2}{3}\end{bmatrix}}{\theta\begin{bmatrix}1\\\frac{2}{3}\end{bmatrix}} = -\frac{1}{12}\frac{\left\{\theta'\begin{bmatrix}1\\1\end{bmatrix}\right\}^2}{\theta^2\begin{bmatrix}1\\0\end{bmatrix}\theta^2\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}\theta^6\begin{bmatrix}1\\\frac{2}{3}\end{bmatrix}}\left(\theta^8\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix} - 10\theta^4\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}\theta^4\begin{bmatrix}1\\\frac{2}{3}\end{bmatrix} + 9\theta^8\begin{bmatrix}1\\\frac{2}{3}\end{bmatrix} \right)
$$

and

$$
\frac{\theta''\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}}{\theta\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}} - \frac{\theta''\begin{bmatrix}1\\\frac{2}{3}\end{bmatrix}}{\theta\begin{bmatrix}1\\\frac{2}{3}\end{bmatrix}} = -\frac{1}{12}\frac{\left\{\theta'\begin{bmatrix}1\\1\end{bmatrix}\right\}^2\theta\begin{bmatrix}1\\0\end{bmatrix}}{\theta^2\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}\theta^5\begin{bmatrix}1\\\frac{2}{3}\end{bmatrix}}\left(\theta^4\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix} - 9\theta^4\begin{bmatrix}1\\\frac{2}{3}\end{bmatrix} \right).
$$

Proof. Comparing the coefficients of the term z^2 in equation (3.2), we have

$$
\frac{\theta''\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}}{\theta\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}} - \frac{\theta''\begin{bmatrix}1\\\frac{2}{3}\end{bmatrix}}{\theta\begin{bmatrix}1\\\frac{2}{3}\end{bmatrix}} = \left\{\frac{\theta'\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}}{\theta\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}}\right\}^2 - \left\{\frac{\theta'\begin{bmatrix}1\\\frac{2}{3}\end{bmatrix}}{\theta\begin{bmatrix}1\\\frac{2}{3}\end{bmatrix}}\right\}^2 + \left\{\theta'\begin{bmatrix}1\\1\end{bmatrix}\right\}^2\frac{\theta\begin{bmatrix}1\\0\end{bmatrix}\theta\begin{bmatrix}1\\\frac{2}{3}\end{bmatrix}}{\theta^2\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}\theta^2\begin{bmatrix}1\\\frac{2}{3}\end{bmatrix}}.
$$

13

From equation (4.2), it follows that

$$\frac{\theta''\left[\begin{smallmatrix}1\\\frac{1}{3}\end{smallmatrix}\right]}{\theta\left[\begin{smallmatrix}1\\\frac{1}{3}\end{smallmatrix}\right]} - \frac{\theta''\left[\begin{smallmatrix}1\\\frac{2}{3}\end{smallmatrix}\right]}{\theta\left[\begin{smallmatrix}1\\\frac{2}{3}\end{smallmatrix}\right]}$$

$$= -\frac{1}{12}\frac{\left\{\theta'\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\right\}^2}{\theta^2\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\theta^2\left[\begin{smallmatrix}1\\\frac{1}{3}\end{smallmatrix}\right]\theta^6\left[\begin{smallmatrix}1\\\frac{2}{3}\end{smallmatrix}\right]}\left(\theta^8\left[\begin{smallmatrix}1\\\frac{1}{3}\end{smallmatrix}\right] + 2\theta^4\left[\begin{smallmatrix}1\\\frac{1}{3}\end{smallmatrix}\right]\theta^4\left[\begin{smallmatrix}1\\\frac{2}{3}\end{smallmatrix}\right] - 3\theta^8\left[\begin{smallmatrix}1\\\frac{2}{3}\end{smallmatrix}\right] - 12\theta^3\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\theta^5\left[\begin{smallmatrix}1\\\frac{2}{3}\end{smallmatrix}\right]\right).$$

The first formula follows from Proposition 7.1.

The second formula follows from the fact that

$$\left(\theta^8\left[\begin{smallmatrix}1\\\frac{1}{3}\end{smallmatrix}\right] - 10\theta^4\left[\begin{smallmatrix}1\\\frac{1}{3}\end{smallmatrix}\right]\theta^4\left[\begin{smallmatrix}1\\\frac{2}{3}\end{smallmatrix}\right] + 9\theta^8\left[\begin{smallmatrix}1\\\frac{2}{3}\end{smallmatrix}\right]\right) = \left(\theta^4\left[\begin{smallmatrix}1\\\frac{1}{3}\end{smallmatrix}\right] - \theta^4\left[\begin{smallmatrix}1\\\frac{2}{3}\end{smallmatrix}\right]\right)\left(\theta^4\left[\begin{smallmatrix}1\\\frac{1}{3}\end{smallmatrix}\right] - 9\theta^4\left[\begin{smallmatrix}1\\\frac{2}{3}\end{smallmatrix}\right]\right)$$

$$= \theta^3\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}1\\\frac{2}{3}\end{smallmatrix}\right]\left(\theta^4\left[\begin{smallmatrix}1\\\frac{1}{3}\end{smallmatrix}\right] - 9\theta^4\left[\begin{smallmatrix}1\\\frac{2}{3}\end{smallmatrix}\right]\right).$$

$\square$

7.2   Derivation of a Riccati equation

Theorem 7.3. *For every* $\tau \in \mathbb{H}^2$, *set*

$$W = \theta^4\left[\begin{smallmatrix}1\\\frac{1}{3}\end{smallmatrix}\right](0,\tau)/\theta^4\left[\begin{smallmatrix}1\\\frac{2}{3}\end{smallmatrix}\right](0,\tau) = \left\{\sqrt{3}\frac{(q^2;q^2)_\infty(q^3;q^3)_\infty^2}{(q;q)_\infty^2(q^6;q^6)_\infty}\right\}^4, \quad q = \exp(2\pi i\tau).$$

Then, W satisfies the following Riccati equation:

$$q\frac{d}{dq}W = \frac{1}{2^3}\left\{\frac{(q;q)_\infty^2(q^3;q^3)_\infty^2}{(q^2;q^2)_\infty(q^6;q^6)_\infty}\right\}^2\left(W^2 - 10W + 9\right).$$

Proof. The heat equation (2.6) and Proposition 7.2 imply that

$$4\pi i\frac{d}{d\tau}\log\theta\left[\begin{smallmatrix}1\\\frac{1}{3}\end{smallmatrix}\right] - 4\pi i\frac{d}{d\tau}\log\theta\left[\begin{smallmatrix}1\\\frac{2}{3}\end{smallmatrix}\right]$$

$$= -\left\{\theta'\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\right\}^2\frac{\left\{\theta^8\left[\begin{smallmatrix}1\\\frac{1}{3}\end{smallmatrix}\right] - 10\theta^4\left[\begin{smallmatrix}1\\\frac{1}{3}\end{smallmatrix}\right]\theta^4\left[\begin{smallmatrix}1\\\frac{2}{3}\end{smallmatrix}\right] + 9\theta^8\left[\begin{smallmatrix}1\\\frac{2}{3}\end{smallmatrix}\right]\right\}}{12\theta^2\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\theta^2\left[\begin{smallmatrix}1\\\frac{1}{3}\end{smallmatrix}\right]\theta^6\left[\begin{smallmatrix}1\\\frac{2}{3}\end{smallmatrix}\right]},$$

14

which shows that

$$4\pi i \frac{d}{d\tau} \log \theta^4 \begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix} - 4\pi i \frac{d}{d\tau} \log \theta^4 \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix}$$

$$= - \left\{ \theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}^2 \frac{\left\{ \theta^8 \begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix} - 10\theta^4 \begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix} \theta^4 \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix} + 9\theta^8 \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix} \right\}}{3\theta^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \theta^2 \begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix} \theta^6 \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix}}.$$

Moreover, Jacobi's triple product identity (2.5) yields

$$\frac{\left\{ \theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}^2 \theta^2 \begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix}}{\theta^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \theta^2 \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix}} = \left\{ -\sqrt{3}\pi \prod_{n=1}^{\infty} \frac{(1-q^n)^2 (1-q^{3n})^2}{(1-q^{2n})(1-q^{6n})} \right\}^2 = \left\{ -\sqrt{3}\pi \frac{\eta^2(\tau)\eta^2(3\tau)}{\eta(2\tau)\eta(6\tau)} \right\}^2.$$

Setting

$$(X, Y) = \left(\theta^4 \begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix}, \theta^4 \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix} \right),$$

we obtain

$$\frac{dX}{d\tau} Y - X \frac{dY}{d\tau} = \frac{2\pi i}{2^3} \left\{ \frac{\eta^2(\tau)\eta^2(3\tau)}{\eta(2\tau)\eta(6\tau)} \right\}^2 \left(X^2 - 10XY + 9Y^2 \right), \qquad (7.2)$$

which implies that

$$\frac{d}{d\tau} W = \frac{d}{d\tau} \left(\frac{X}{Y} \right) = \frac{\frac{dX}{d\tau} Y - X \frac{dY}{d\tau}}{Y^2} = \frac{2\pi i}{2^3} \left\{ \frac{\eta^2(\tau)\eta^2(3\tau)}{\eta(2\tau)\eta(6\tau)} \right\}^2 \left(W^2 - 10W + 9 \right).$$

The theorem can be obtained by considering that $\dfrac{d}{d\tau} = \dfrac{dq}{d\tau} \dfrac{d}{dq} = 2\pi i q \dfrac{d}{dq}$. $\qquad\square$

By the second formula of Proposition 7.2, we obtain the following theorem.

Theorem 7.4. *For every $\tau \in \mathbb{H}^2$, set*

$$W = \left\{ \frac{\eta(2\tau)\eta^2(3\tau)}{\eta^2(\tau)\eta(6\tau)} \right\}^4 = \left\{ \frac{(q^2; q^2)_\infty (q^3; q^3)_\infty^2}{(q; q)_\infty^2 (q^6; q^6)_\infty} \right\}^4, \quad q = \exp(2\pi i \tau).$$

Then, W satisfies the following differential equation:

$$q \frac{d}{dq} W = \frac{(q^2; q^2)_\infty^7 (q^3; q^3)_\infty^7}{(q; q)_\infty^5 (q^6; q^6)_\infty^5} (W - 1), \ \ or \ \ \frac{d}{d\tau} W = 2\pi i \frac{\eta^7(2\tau)\eta^7(3\tau)}{\eta^5(\tau)\eta^5(6\tau)} (W - 1).$$

15

8 ODE satisfied by a modular form of level 8

8.1 Preliminary results

From Matsuda [5], recall the following theta constant identities.

Theorem 8.1. *For every $\tau \in \mathbb{H}^2$, we have*

$$\theta\begin{bmatrix} 1 \\ 0 \end{bmatrix}\theta^3\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} - \theta\begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix}\theta^3\begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} - \theta\begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix}\theta^3\begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} = 0, \tag{8.1}$$

$$\theta^2\begin{bmatrix} 1 \\ 0 \end{bmatrix}\theta\begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix}\theta\begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} - \theta^2\begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix}\theta^2\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} + \theta^2\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}\theta^2\begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} = 0, \tag{8.2}$$

$$\theta^4\begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} - \theta^4\begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} - \theta\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}\theta^3\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0. \tag{8.3}$$

From these theta constant identities, we can obtain the following proposition.

Proposition 8.2. *For every $\tau \in \mathbb{H}^2$, we have*

$$\frac{\theta''\begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix}}{\theta\begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix}} - \frac{\theta''\begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix}}{\theta\begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix}} = -\frac{1}{8}\frac{\left\{\theta'\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}^2\theta\begin{bmatrix} 1 \\ 0 \end{bmatrix}\theta\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}}{\theta^4\begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix}\theta^4\begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix}}\left(\theta^4\begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} - 6\theta^2\begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix}\theta^2\begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} + \theta^4\begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix}\right).$$

Proof. By Lemma 2.1, we first note that

$$\theta^2\begin{bmatrix} 0 \\ 0 \end{bmatrix}(0, \tau) = \theta^2\begin{bmatrix} 0 \\ 0 \end{bmatrix}(0, 2\tau) + \theta^2\begin{bmatrix} 1 \\ 0 \end{bmatrix}(0, 2\tau),$$

$$\theta^2\begin{bmatrix} 0 \\ 1 \end{bmatrix}(0, \tau) = \theta^2\begin{bmatrix} 0 \\ 0 \end{bmatrix}(0, 2\tau) - \theta^2\begin{bmatrix} 1 \\ 0 \end{bmatrix}(0, 2\tau),$$

$$\theta^2\begin{bmatrix} 0 \\ 1 \end{bmatrix}(0, 2\tau) = \theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(0, \tau)\theta\begin{bmatrix} 0 \\ 1 \end{bmatrix}(0, \tau).$$

Comparing the coefficients of the term z^2 in equation (3.3) yields

$$\frac{\theta''\begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix}}{\theta\begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix}} - \frac{\theta''\begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix}}{\theta\begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix}} = \left\{\frac{\theta'\begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix}}{\theta\begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix}}\right\}^2 - \left\{\frac{\theta'\begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix}}{\theta\begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix}}\right\}^2 + \left\{\theta'\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}^2\frac{\theta\begin{bmatrix} 1 \\ 0 \end{bmatrix}\theta\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}}{\theta^2\begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix}\theta^2\begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix}}. \tag{8.4}$$

Equations (4.3), (4.4), (8.1), and (8.3) imply that

$$
\frac{\theta''\left[\begin{smallmatrix}1\\\frac{1}{4}\end{smallmatrix}\right]}{\theta\left[\begin{smallmatrix}1\\\frac{1}{4}\end{smallmatrix}\right]}-\frac{\theta''\left[\begin{smallmatrix}1\\\frac{3}{4}\end{smallmatrix}\right]}{\theta\left[\begin{smallmatrix}1\\\frac{3}{4}\end{smallmatrix}\right]}=\frac{1}{2}\left\{\theta'\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\right\}^2\frac{\theta\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}1\\\frac{1}{2}\end{smallmatrix}\right]}{\theta^2\left[\begin{smallmatrix}1\\\frac{1}{4}\end{smallmatrix}\right]\theta^2\left[\begin{smallmatrix}1\\\frac{3}{4}\end{smallmatrix}\right]}\left\{\frac{\theta^2\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right](0,2\tau)+\theta^2\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right](0,2\tau)}{\theta^2\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right](0,2\tau)}+2\right\}
$$

$$
-\frac{1}{4}\left\{\theta'\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\right\}^2\frac{\theta^3\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\theta^7\left[\begin{smallmatrix}1\\\frac{1}{2}\end{smallmatrix}\right]}{\theta^6\left[\begin{smallmatrix}1\\\frac{1}{4}\end{smallmatrix}\right]\theta^6\left[\begin{smallmatrix}1\\\frac{3}{4}\end{smallmatrix}\right]},
$$

which shows

$$
\frac{\theta''\left[\begin{smallmatrix}1\\\frac{1}{4}\end{smallmatrix}\right]}{\theta\left[\begin{smallmatrix}1\\\frac{1}{4}\end{smallmatrix}\right]}-\frac{\theta''\left[\begin{smallmatrix}1\\\frac{3}{4}\end{smallmatrix}\right]}{\theta\left[\begin{smallmatrix}1\\\frac{3}{4}\end{smallmatrix}\right]}=\left\{\theta'\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\right\}^2\frac{\theta\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}1\\\frac{1}{2}\end{smallmatrix}\right]}{\theta^2\left[\begin{smallmatrix}1\\\frac{1}{4}\end{smallmatrix}\right]\theta^2\left[\begin{smallmatrix}1\\\frac{3}{4}\end{smallmatrix}\right]}\cdot\frac{\theta^2\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right](0,4\tau)}{\theta^2\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right](0,2\tau)}+
$$

$$
+\left\{\theta'\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\right\}^2\frac{\theta\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}1\\\frac{1}{2}\end{smallmatrix}\right]}{\theta^2\left[\begin{smallmatrix}1\\\frac{1}{4}\end{smallmatrix}\right]\theta^2\left[\begin{smallmatrix}1\\\frac{3}{4}\end{smallmatrix}\right]}-\frac{1}{4}\left\{\theta'\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\right\}^2\frac{\theta^3\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\theta^7\left[\begin{smallmatrix}1\\\frac{1}{2}\end{smallmatrix}\right]}{\theta^6\left[\begin{smallmatrix}1\\\frac{1}{4}\end{smallmatrix}\right]\theta^6\left[\begin{smallmatrix}1\\\frac{3}{4}\end{smallmatrix}\right]}.
$$

Jacobi's triple product identity (2.5) yields

$$
\left\{\theta'\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\right\}^2\frac{\theta\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}1\\\frac{1}{2}\end{smallmatrix}\right]}{\theta^2\left[\begin{smallmatrix}1\\\frac{1}{4}\end{smallmatrix}\right]\theta^2\left[\begin{smallmatrix}1\\\frac{3}{4}\end{smallmatrix}\right]}\cdot\frac{\theta^2\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right](0,4\tau)}{\theta^2\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right](0,2\tau)}=\frac{1}{8}\left\{\theta'\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\right\}^2\frac{\theta^3\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\theta^7\left[\begin{smallmatrix}1\\\frac{1}{2}\end{smallmatrix}\right]}{\theta^6\left[\begin{smallmatrix}1\\\frac{1}{4}\end{smallmatrix}\right]\theta^6\left[\begin{smallmatrix}1\\\frac{3}{4}\end{smallmatrix}\right]}
$$

$$
=4\pi^2\sqrt{2}\,\frac{\eta^{13}(4\tau)}{\eta^2(\tau)\eta(2\tau)\eta^6(8\tau)}.
$$

Therefore, it follows that

$$
\frac{\theta''\left[\begin{smallmatrix}1\\\frac{1}{4}\end{smallmatrix}\right]}{\theta\left[\begin{smallmatrix}1\\\frac{1}{4}\end{smallmatrix}\right]}-\frac{\theta''\left[\begin{smallmatrix}1\\\frac{3}{4}\end{smallmatrix}\right]}{\theta\left[\begin{smallmatrix}1\\\frac{3}{4}\end{smallmatrix}\right]}=-\frac{1}{8}\left\{\theta'\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\right\}^2\frac{\theta^3\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\theta^7\left[\begin{smallmatrix}1\\\frac{1}{2}\end{smallmatrix}\right]}{\theta^6\left[\begin{smallmatrix}1\\\frac{1}{4}\end{smallmatrix}\right]\theta^6\left[\begin{smallmatrix}1\\\frac{3}{4}\end{smallmatrix}\right]}+\left\{\theta'\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\right\}^2\frac{\theta\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}1\\\frac{1}{2}\end{smallmatrix}\right]}{\theta^2\left[\begin{smallmatrix}1\\\frac{1}{4}\end{smallmatrix}\right]\theta^2\left[\begin{smallmatrix}1\\\frac{3}{4}\end{smallmatrix}\right]}.
$$
$$(8.5)$$

17

Equation (8.1) implies that

$$\frac{\theta''\begin{bmatrix}1\\\frac{1}{4}\end{bmatrix}}{\theta\begin{bmatrix}1\\\frac{1}{4}\end{bmatrix}}-\frac{\theta''\begin{bmatrix}1\\\frac{3}{4}\end{bmatrix}}{\theta\begin{bmatrix}1\\\frac{3}{4}\end{bmatrix}}=-\frac{1}{8}\frac{\left\{\theta'\begin{bmatrix}1\\1\end{bmatrix}\right\}^2\theta\begin{bmatrix}1\\0\end{bmatrix}\theta\begin{bmatrix}1\\\frac{1}{2}\end{bmatrix}}{\theta^4\begin{bmatrix}1\\\frac{1}{4}\end{bmatrix}\theta^4\begin{bmatrix}1\\\frac{3}{4}\end{bmatrix}}\left(\theta^4\begin{bmatrix}1\\\frac{1}{4}\end{bmatrix}-6\theta^2\begin{bmatrix}1\\\frac{1}{4}\end{bmatrix}\theta^2\begin{bmatrix}1\\\frac{3}{4}\end{bmatrix}+\theta^4\begin{bmatrix}1\\\frac{3}{4}\end{bmatrix}\right).$$

$\square$

Corollary 8.3. *For every* $\tau \in \mathbb{H}^2$, *we have*

$$\frac{\eta^2(\tau)\eta(2\tau)\eta^3(4\tau)}{\eta^2(8\tau)}=1-2\sum_{n=1}^{\infty}\left(\sum_{d|n}d\left(\frac{8}{d}\right)\right)q^n,$$

where $q = \exp(2\pi i\tau)$ *and for each* $m \in \mathbb{N}$,

$$\left(\frac{8}{m}\right)=\begin{cases}+1, & \text{if } m \equiv \pm 1 \ (\mathrm{mod}\,8),\\ -1, & \text{if } m \equiv \pm 3 \ (\mathrm{mod}\,8),\\ 0, & \text{if } m \equiv 0 \ (\mathrm{mod}\,2).\end{cases}$$

Proof. By equation (8.4), we have

$$\left\{\theta'\begin{bmatrix}1\\1\end{bmatrix}\right\}^2\frac{\theta\begin{bmatrix}1\\0\end{bmatrix}\theta\begin{bmatrix}1\\\frac{1}{2}\end{bmatrix}}{\theta^2\begin{bmatrix}1\\\frac{1}{4}\end{bmatrix}\theta^2\begin{bmatrix}1\\\frac{3}{4}\end{bmatrix}}=\frac{d^2}{dz^2}\log\theta\begin{bmatrix}1\\\frac{1}{4}\end{bmatrix}(z)\Big|_{z=0}-\frac{d^2}{dz^2}\log\theta\begin{bmatrix}1\\\frac{3}{4}\end{bmatrix}(z)\Big|_{z=0}.$$

The corollary follows from Jacobi's triple product identity (2.5). $\square$

Corollary 8.4. *For every* $\tau \in \mathbb{H}^2$, *we have*

$$\frac{\eta^{13}(4\tau)}{\eta^2(\tau)\eta(2\tau)\eta^6(8\tau)}=1-2\sum_{n=1}^{\infty}\left(\sum_{d|n}\left(d\left(\frac{8}{d}\right)-2\frac{n}{d}\left(\frac{8}{d}\right)\right)\right)q^n,$$

where $q = \exp(2\pi i\tau)$.

Proof. The heat equation (2.6) and equation (8.5) imply that

$$\frac{d}{d\tau}\log\frac{\theta\begin{bmatrix}1\\\frac{1}{4}\end{bmatrix}}{\theta\begin{bmatrix}1\\\frac{3}{4}\end{bmatrix}}=\frac{2\pi i}{\sqrt{2}}\left\{\frac{\eta^{13}(4\tau)}{\eta^2(\tau)\eta(2\tau)\eta^6(8\tau)}-\frac{\eta^2(\tau)\eta(2\tau)\eta^3(4\tau)}{\eta^2(8\tau)}\right\},$$

18

which shows that

$$\frac{\eta^{13}(4\tau)}{\eta^2(\tau)\eta(2\tau)\eta^6(8\tau)} - \frac{\eta^2(\tau)\eta(2\tau)\eta^3(4\tau)}{\eta^2(8\tau)} = 4\sum_{n=1}^{\infty}\left(\sum_{d|n}\frac{n}{d}\left(\frac{8}{d}\right)\right)q^n.$$

$\square$

8.2   Derivation of a Riccati equation

Theorem 8.5. *For every $\tau \in \mathbb{H}^2$, set*

$$W = \theta^2\left[\begin{array}{c}1\\\frac{1}{4}\end{array}\right](0,\tau)/\theta^2\left[\begin{array}{c}1\\\frac{3}{4}\end{array}\right](0,\tau), \quad q = \exp(2\pi i \tau).$$

Then, W satisfies the following Riccati equation:

$$q\frac{dW}{dq} = \frac{1}{(\sqrt{2})^5}\frac{(q;q)_\infty^2(q^2;q^2)_\infty(q^4;q^4)_\infty^3}{(q^8;q^8)_\infty^2}\left(W^2 - 6W + 1\right).$$

Proof. Set

$$(X,Y) = \left(\theta^2\left[\begin{array}{c}1\\\frac{1}{4}\end{array}\right], \theta^2\left[\begin{array}{c}1\\\frac{3}{4}\end{array}\right]\right).$$

The heat equation (2.6) and Proposition 8.2 imply that

$$4\pi i\frac{d}{d\tau}\log\theta\left[\begin{array}{c}1\\\frac{1}{4}\end{array}\right] - 4\pi i\frac{d}{d\tau}\log\theta\left[\begin{array}{c}1\\\frac{3}{4}\end{array}\right] = -\frac{1}{8}\left\{\theta'\left[\begin{array}{c}1\\1\end{array}\right]\right\}^2\frac{\theta\left[\begin{array}{c}1\\0\end{array}\right]\theta\left[\begin{array}{c}1\\\frac{1}{2}\end{array}\right]}{\theta^4\left[\begin{array}{c}1\\\frac{1}{4}\end{array}\right]\theta^4\left[\begin{array}{c}1\\\frac{3}{4}\end{array}\right]}(X^2-6XY+Y^2),$$

which shows that

$$4\pi i\frac{d}{d\tau}\log X - 4\pi i\frac{d}{d\tau}\log Y = -\frac{1}{4}\left\{\theta'\left[\begin{array}{c}1\\1\end{array}\right]\right\}^2\frac{\theta\left[\begin{array}{c}1\\0\end{array}\right]\theta\left[\begin{array}{c}1\\\frac{1}{2}\end{array}\right]}{\theta^4\left[\begin{array}{c}1\\\frac{1}{4}\end{array}\right]\theta^4\left[\begin{array}{c}1\\\frac{3}{4}\end{array}\right]}(X^2 - 6XY + Y^2).$$

Moreover, Jacobi's triple product identity (2.5) yields

$$\left\{\theta'\left[\begin{array}{c}1\\1\end{array}\right]\right\}^2\frac{\theta\left[\begin{array}{c}1\\0\end{array}\right]\theta\left[\begin{array}{c}1\\\frac{1}{2}\end{array}\right]}{\theta^2\left[\begin{array}{c}1\\\frac{1}{4}\end{array}\right]\theta^2\left[\begin{array}{c}1\\\frac{3}{4}\end{array}\right]} = 4\sqrt{2}\pi^2\prod_{n=1}^{\infty}\frac{(1-q^n)^2(1-q^{2n})(1-q^{4n})^3}{(1-q^{8n})^2} = 4\sqrt{2}\pi^2\frac{\eta^2(\tau)\eta(2\tau)\eta^3(4\tau)}{\eta^2(8\tau)}.$$

19

Therefore, it follows that

$$\frac{dX}{d\tau}Y - X\frac{dY}{d\tau} = \frac{2\pi i}{4\sqrt{2}}\frac{\eta^2(\tau)\eta(2\tau)\eta^3(4\tau)}{\eta^2(8\tau)}\left(X^2 - 6XY + Y^2\right), \qquad (8.6)$$

which implies that

$$\frac{d}{d\tau}W = \frac{d}{d\tau}\left(\frac{X}{Y}\right) = \frac{\frac{dX}{d\tau}Y - X\frac{dY}{d\tau}}{Y^2} = \frac{2\pi i}{4\sqrt{2}}\frac{\eta^2(\tau)\eta(2\tau)\eta^3(4\tau)}{\eta^2(8\tau)}\left(W^2 - 6W + 1\right).$$

The theorem can be obtained by considering that $\dfrac{d}{d\tau} = \dfrac{dq}{d\tau}\dfrac{d}{dq} = 2\pi i q\dfrac{d}{dq}$. $\qquad\square$

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